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NONLINEAR DEFORMATION AND BUCKLING OF CURVILINEAR PIPES LOADED BY EXTERNAL PRESSURE

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The problem of loading of a thin-walled elastic pipe (a toroidal shell) by external pressure is examined in a geometrically nonlinear formulation. A numerical algorithm is used to study the nonlinear deformation of the shell and the stability of its equilibrium states when its cross section has undergone a significant change in shape. Results are presented from a determination of the critical stresses of curvilinear pipes with allowance for moments in the subcritical state. These results are compared with the approximate solution.

The linear stability of long curvilinear pipes of circular cross section under external pressure was examined in [1, 2], where approximate solutions of the problem were obtained. The solutions make it possible to determine the critical pressure as a function of the geometric characteristics of the pipe. The solutions were constructed without consideration of subcritical deformation of the cross sections, and it was assumed that the radius of the cross section was small compared to the radius of curvature of the axial line of the pipe.

It is of interest to examine the problem of the stability of pipes in a geometrically nonlinear formulation with allowance for moments in the subcritical stress-strain state, evaluate the effect of geometric parameters on the critical load, and study the equilibrium modes in the region of large displacements.

We shall examine a pipe representing a sector of an elastic toroidal thin-walled shell obeying the Kirchhoff–Love hypotheses. It is assumed [1, 2] that in the deformed state the shell is toroidal and that the cross sections remain planar and normal to the axial line and can undergo deformation in their own planes. Geometrically nonlinear relations were derived in [3, 4] on the basis of these assumptions and a finite-element model of the shell was constructed. The geometry of the cross section is determined by the nodal values of the coordinates and the direction cosines of unit normals to the contour of the cross section.

We shall find the strain state of the shell by the method of discrete continuation, which is based on stepwise determination of the solution and its subsequent refinement by iteration. At each step, we construct a solution which is linearized with respect to the equilibrium state that is found. This solution is then refined on the basis of the orthogonality condition of the vector of the corrections to the vector of the linearized solution [5].

The system used in the Newton–Raphson method for a certain continuation step has the form

$$\mathbf{H}^{k-1}\delta\mathbf{q}^k + \mathbf{w}^{k-1}\delta p^k + \mathbf{g}^{k-1} = 0, \quad \delta\mathbf{q}^t = \delta[x_{11}, x_{21}, \varphi_1, \ldots, x_{1n}, x_{2n}, \varphi_n, \varepsilon, æ], \tag{1}$$

where $\mathbf{g}$ and $\mathbf{H}$ are the Hess gradient and matrix, which are composed of coefficients of the first and second variations of the total potential energy of the set of finite elements [4], the components of the vector $\mathbf{w}$ are determined by the formula $w_i = \partial^2 U/\partial q_i \partial p$, where U is the force potential associated with the surface pressure p, which is an unknown in the problem, x_{ij} are the coordinates of the nodes of the deformed cross section, φ_j are the angles of rotation of the normal vector in the plane of the cross section, ε and $æ$ are the strain and curvature of the axial line of the shell, n is the number of nodes, and the superscript denotes the iteration number.

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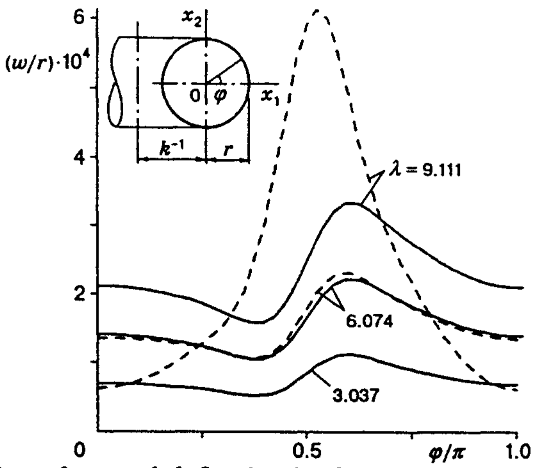

Fig. 1. Distribution of normal deflection in the cross section of a shell: the solid and dashed curves refer to the linear and nonlinear solutions, respectively.

System (1) is supplemented by the control equations [5]

$$(\delta \mathbf{q}^1)^t\,\delta \mathbf{q}^1 + (\delta p^1)^2 = \delta s^2 \ \text{ for } k = 1, \qquad (\delta \mathbf{q}^k)^t\,\delta \mathbf{q}^1 + \delta p^k \delta p^1 = 0 \ \text{ for } k > 1, \tag{2}$$

where δs is the assigned continuation step.

The solution of system (1) is represented in the form

$$\delta \mathbf{q}^k = \delta p^k \mathbf{u}^k + \mathbf{v}^k, \tag{3}$$

where $\mathbf{u}^k$ and $\mathbf{v}^k$ satisfy the systems

$$\mathbf{H}^{k-1}\mathbf{u}^k + \mathbf{w}^{k-1} = 0, \quad \mathbf{H}^{k-1}\mathbf{v}^k + \mathbf{g}^{k-1} = 0. \tag{4}$$

Substitution of (3) into (2) with allowance for $\mathbf{g}^0 = 0$ and $\mathbf{v}^1 = 0$ leads to formulas for calculation of the increment of load

$$\delta p^1 = \pm \delta s / (1 + (\mathbf{u}^1)^t \mathbf{u}^1)^{1/2} \ \text{ for } k = 1, \qquad \delta p^k = -(\mathbf{v}^k)^t \mathbf{u}^1 / (1 + (\mathbf{u}^k)^t \mathbf{u}^1) \ \text{ for } k > 1. \tag{5}$$

The sign in (5) determines the direction of continuation of the solution and at the ith step is chosen on the basis of the following condition (no summation is performed over i)

$$(\delta \mathbf{q}_i^1)^t \delta \mathbf{q}_{i-1}^1 + \delta p_i^1 \delta p_{i-1}^1 > 0.$$

After each iteration, the new values of the desired parameters are calculated from the formulas

$$x_{ij}^{k+1} = x_{ij}^k + \delta x_{ij}^{k+1}, \qquad (\lambda_{ij}^n)^{k+1} = (\lambda_{ij}^n)^k \cos \delta\varphi_j^{k+1} + \lambda_{ij}^k \sin \delta\varphi_j^{k+1},$$

$$\varepsilon^{k+1} = \varepsilon^k + \delta\varepsilon^{k+1}, \qquad æ^{k+1} = æ^k + \delta æ^{k+1}, \qquad p^{k+1} = p^k + \delta p^{k+1},$$

where λ_{ij}^n and λ_{ij} are the direction cosines of the normal and tangent unit vectors to the contour of the cross section of the shell.

The question of the stability of the equilibrium states that are found is answered by checking to see if the matrix $\mathbf{H}^0$ has a fixed sign. This information can be obtained by using the direct Gauss method in the solution of systems (4).

We shall study the deformation of a toroidal shell having the following characteristics: $r/h = 100$, $kr = 0{,}5$, $\nu = 0{,}3$, and $\mu = 165{,}2$, where r is the radius of the cross section, k is the curvature of the axial line, h is the wall thickness; ν is the Poisson ratio, and $\mu = \sqrt{12(1-\nu^2)}\, kr^2/h$ is a parameter that characterizes the initial curvature. We shall examine equilibrium modes that are symmetric relative to the plane of curvature of the axial line. There will be no restrictions on the possibility of self-intersection of the surface of the shell.

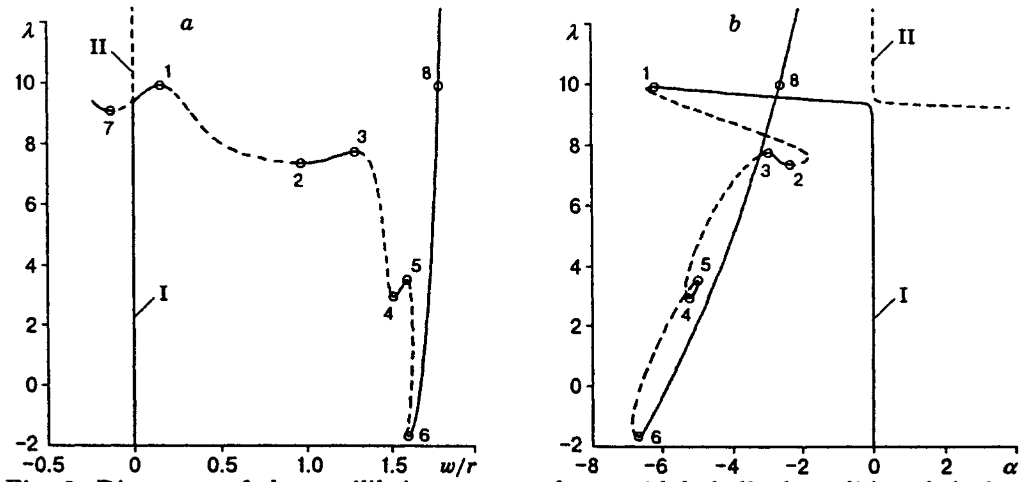

Fig. 2. Diagrams of the equilibrium states of a toroidal shell: the solid and dashed curves refer to the stable and unstable states, respectively.

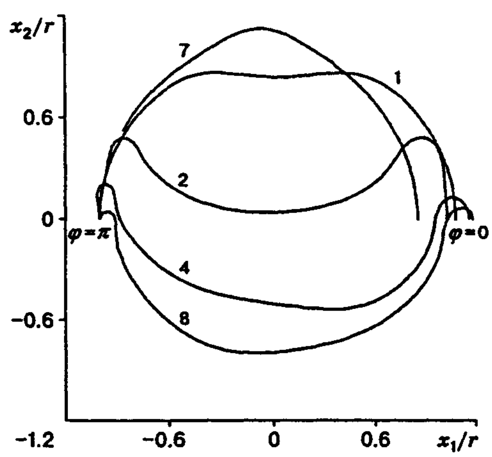

Fig. 3

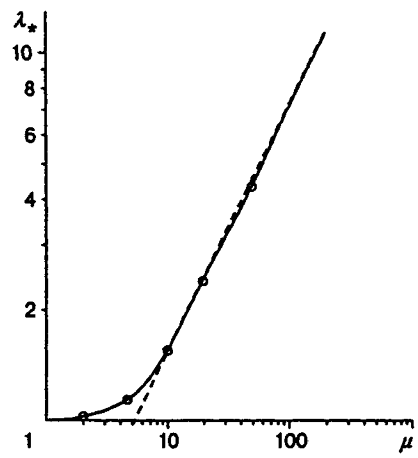

Fig. 4

Fig. 3. Shapes of the cross section of the shell.

Fig. 4. Dependence of the critical pressure parameter of a pipe on its initial curvature parameter: the solid curve shows the numerical solution, the points show the approximate solution in [2], and the dashed curve shows the solution calculated by means of Eq. (6).

Calculations show that the cross section is distorted near the line of zero Gaussian curvature of the shell regardless of how low the pressure level is. Figure 1 shows curves depicting the distribution of the normal deflection w across the shell. The curves were obtained by the above numerical algorithm and on the basis of the BENDPAC program [6] for a linear analysis of toroidal shells. Here $\lambda = p/p_0$, where $p_0 = (1/4)E(1-\nu^2)^{-1}(h/r)^3$ is the critical pressure in the straight pipe. Nonlinear deformation becomes quite evident for $\lambda > 6$: the displacements are redistributed in the cross section and a local indentation is formed. The size of the indentation increases with the load up to $\lambda_* = 9{,}927$, where the shell becomes unstable. Thus, in contrast to straight pipes [7], a toroidal shell becomes unstable when the limit point is reached.

Figure 2 shows diagrams of the equilibrium states. The values plotted off the x axis represent the dimensionless deflection for $\varphi = \pi/2$ (a) and the parameter that characterizes the curvature of the axial line of the shell is $\alpha = \sqrt{12(1-\nu^2)}\ae r^2/h$ (b). The parameter that characterizes the external pressure λ is plotted off the y axis. There are six limit points on the main branch of the nonlinear solution (curves I), which passes through a point representing the initial undeformed state. Figure 3 shows shapes of the cross section which correspond to the equilibrium states in Fig. 2.

We also studied the branch of solutions which is isolated from the main branch (curves II) and has one limit point (point 7 in Fig. 2a). The linear solution of the problem for $\lambda > \lambda_*$ was taken as the initial approximation for the transition to the isolated branch. We note that the shapes of deformation of the cross sections corresponding to the isolated branch (state 7 in Fig. 3) are similar to the modes of flattening of the shell in pure bending by edge moments. In both cases, the cross sections of the shells for which the value of the parameter μ is large are deformed as a result of local bending in the neighborhood of the line of zero Gaussian curvature.

To determine the effect of the geometric characteristics on the stability of curvilinear pipes, we examined shells with values of the parameter $r/h = 50$, 100, and 200 and the Poisson ratio $\nu = 0{,}3$. The calculations showed that changes which occur in r, h, and k with a fixed value of μ have almost no effect on the critical pressure parameter λ_*. Figure 4 shows the dependence of λ_* on the initial curvature parameter μ. It was established that the following approximate relation is valid for $\mu > 10$:

$$\lambda_* = (1/3)\mu^{2/3}. \tag{6}$$

It is noteworthy that the approximate solution found in [2] under several simplifying assumptions is sufficiently accurate in determining the critical pressures within a broad range of geometric parameters of pipes.

Pipes with small values of the curvature parameter $\mu < 10$ also become unstable upon passage through the limit point. In this case, the forms of the cross section in the neighborhood of the critical point are close to an oval whose major axis lies in the plane of curvature of the pipe.

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REFERENCES

1. D. L. Kostovetskii, "Stability of equilibrium of the curve of a thin-walled pipe of circular cross section loaded by external pressure," *Izv. Akad. Nauk SSSR, Otd. Tekh. Nauk, Mekh. Mashinostr.*, No. 1, 177–179 (1961).
2. E. L. Axelrad, "Stability of a curvilinear pipe of circular cross section under external pressure," *Inzh. Zh., Mekh. Tverd. Tela*, No. 2, 170–176 (1967).
3. V. V. Kuznetsov and S. V. Levyakov, "Nonlinear Kàrmàn problem for toroidal shells of arbitrary cross section," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 2, 136–142 (1992).
4. V. V. Kuznetsov and S. V. Levyakov, "Nonlinear bending of toroidal shells of arbitrary cross section loaded by internal pressure," *Prikl. Mekh. Tekh. Fiz.*, **34**, No. 4, 112–118 (1993).
5. Y.-B. Yang and M.-S. Shieh, "Solution method for nonlinear problems with multiple critical points," *AIAA J.*, **28**, 2110–2116 (1990).
6. J. F. Whatham, "Pipe bending analysis by thin shell theory," *Trans. ASME, Ser. E, J. Appl. Mech.*, **53**, No. 3, 173–180 (1986).
7. A. S. Vol'mir, *Stability of Deformable Systems* [in Russian], Nauka, Moscow (1967).